Special invariant subspaces, local-to-global properties and approximation problems

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One or more questions of the following type will be considered.

1. LOCAL-TO-GLOBAL PROPERTIES

If every member of a collection of operators (e.g., a group, semigroup, algebra) has a certain property, then under what conditions does the collection have it simultaneously? A. Simonic (1992) gave an answer to the question: If every member of a group G of matrices is (individually) similar to a positive definite matrix, is G simultaneously similar to a group of positive definite matrices? His answer is no in general, but yes if G is divisible. One question to ponder: change "positive definite" to "positive" or "nonnegative" in the question above. "Nonnegative" means that every entry of the matrix is nonnegative. More generally, an operator on a function space is said to be nonnegative if it takes the set of nonnegative functions into itself. Another question of this type: If each member of a semigroup S of matrices has the increasing spectrum property (see below for definition), is S simultaneously triangularizable by a permutation similarity?

2. STANDARD INVARIANT SUBSPACES

Let L be a space of functions on X (e.g., an L_p space with a given measure on the underlying space X). If Y is a subspace of X, and M the set of all functions supported on Y, then we call M a standard subspace of L. If an operator (or a collection of operators simultaneously) has a nontrivial standard invariant subspace, it is called decomposable. If the standard invariant subspaces for a collection C form a maximal subspace chain, then C is said to have a standard triangularization. We say that an operator T on L has the increasing spectrum property if whenever M and N are standard subspaces of L with $M < N$, then the spectrum of the compression of T to M is contained in the spectrum of its compression to N. (In finite dimensions this means, for a given matrix, that if you take two principal submatrices one contained in the other, then the set of eigenvalues of the smaller submatrix is contained in that of the larger one–which is obviously the case if the matrix is triangular.) Question: If K is a compact operator on an L_2 space and if K has the increasing spectrum property, does K have a standard triangularization? In particular, is K decomposable? L. Marcoux, M. Mastnak, and H. Radjavi (2009) asked the question and showed that the answer is yes if K has finite rank.

3. From Positive to General

Let M be a nonnegative matrix (in the sense of paragraph 1). If the diagonal of M consists exactly of its eigenvalues with the right multiplicities, then M is triangular after a similarity by a permutation. This was extended to infinite-dimensional setting by J. Bernik, L. Marcoux, and H. Radjavi (2012). What about general operators–not necessarily nonnegative? The short answer is easily: no, except in dimension 2. But we are looking for long answers!

4. Almost nilpotent vs nearly nilpotent

An operator $T \in \mathbb{M}_n(\mathbb{C})$ is said to be **almost nilpotent of order** $1 \leq k \leq n$ if $\|T^k\|$ is very small. Let $\mathcal{N}_n^{(k)} = \{N \in \mathbb{M}_n(\mathbb{C}) : N^k = 0\}$. We say that T is **nearly nilpotent of order** k if $\mathrm{dist}(T, \mathcal{N}_n^{(k)})$ is small. Can one show that there exists a function $f : (0, \infty) \to (0, \infty)$, *independent of n*, such that

- $\lim_{x \to 0^+} f(x) = 0$ and
- $\|T^k\|^{1/k} < \varepsilon$ implies $\mathrm{dist}(T, \mathcal{N}_n^{(k)}) < f(\varepsilon)$?

That is, can one show that every almost nilpotent operator of order k is nearly nilpotent of order k? It would still be very, very interesting if one could show that there exists a second function $\mu : \mathbb{N} \to \mathbb{N}$ *independent of n* so that

$$\|T^k\|^{1/k} < \varepsilon \quad \text{implies that} \quad \mathrm{dist}(T, \mathcal{N}_n^{(\mu(k))}) < f(\varepsilon).$$

A positive answer to this question would resolve a question from the PhD Thesis of Lawrence Williams (circa 1976): if $Q \in \mathcal{B}(\ell_2)$ is quasidiagonal and quasinilpotent, is Q a limit of block-diagonal nilpotents? (Definitions available upon request.)

5. Extremely non-normal operators

We shall say that an operator $T \in \mathbb{M}_n(\mathbb{C})$ is **extremely non-normal** if $[T, T^*] := TT^* - T^*T$ is invertible (and thus, in some sense, as "far away" from normal operators as possible). For example, when $n = 2$, then every operator is either normal or extremely non-normal. (Upper-triangularize T and calculate the determinant of $[T^*, T]$.) This fails when $n = 3$, since $[1] \oplus E_{12}$ is neither normal nor extremely non-normal, where $E_{12} \in \mathbb{M}_2(\mathbb{C})$ denotes the usual $(1, 2)$-matrix unit. Can one characterize extremely non-normal operators in $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$?